The Structures and the Interactions of Soliton in two (2+1)-dimensional KdV-type Equations

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Exact solutions in two (2+1)-dimensional KdV-type (Sawada-Kodera and Boussinesq) equations are presented by using the bilinear method. The N-breather solution, the solution to describe the interaction between a line soliton and a *y*-periodic soliton, and the solution to express the interaction between two *y*-periodic solitons are included in our results. Detailed behavior of interactions between a line soliton and a *y*-periodic soliton for the SK equation and between two *y*-periodic solitons for the BS equation are illustrated both analytically and graphically. For these two equations, we only discuss the repulsive interaction keeping the shapes of the soliton unchanged.

Key words: Interactions between two Solitons; Sawada-Kotera Equation; Boussinesq Equation.

1. Introduction

The resent development of the nonlinear wave theory clarifies the role of the ‘soliton’ in various systems [1 – 3]. Solitons are stable, and the interaction between them affects only the phase shifts [1]. Therefore solitons are regarded as the fundamental structures of nonlinear integrable systems. The spatial structures of solitons have usually the solitary wave form whose amplitude tends to zero as $x \longrightarrow \pm\infty$, or the kink form whose amplitude tends to two different constants as $x \longrightarrow \pm\infty$. Because one-dimensional soliton equations are well studied both theoretically aspect and experimentally [2], usually we consider the one-dimensional problem first. Although this simple assumption holds in many physical phenomena, especially in some well-controlled laboratory experiments, real physical phenomena are not regarded strictly as one-dimensional ones and may have an essentially two- or three-dimensional nature. In addition to this, the one-dimensional structures realized initially may be unstable to transverse (two-dimensional) disturbances. The dynamics of nonlinear waves in higher-dimensional space are not understood well enough in comparison with that in one space dimension. One of the main physical reasons is that various localized structures in higher-dimensional space may be considered as candidates for solitons. In previous studies the solutions usually obtained are limited to plane waves or line-solitons that are essentially one-dimensional. This happens even in the case of higher-dimensional soliton equations. This type of solitons is only one possibility of the structures in higher-dimensional space. To clarify the dynamics of higher-dimensional systems we must find possible spatial structures. We expect that the complicated structures typical for higher dimensionality may contribute to the variety of the dynamics. Especially, the interactions would be qualitatively different from those in one-dimensional systems.

Motivated by the above physical reasons, we consider the (2+1)-dimensional Sawada-Kotera (SK) equation [4]

$$u_t + (u_{xxxx} + 5uu_{xx} + \frac{5}{3}u^3 + 5u_{xy})_x - 5\int u_{yy}dx + 5uu_y + 5u_x \int u_y\,\mathrm{d}x = 0 \qquad (1)$$

and the (2+1)-dimensional Boussinesq equation [5]

$$4u_{xy} - 3u_{tt} - au_{xx} + u_{xxxx} + 6(uu_x)_x = 0. \qquad (2)$$

The reduced equation of (1) was first derived by K. Sawada, P. J. Caudry, R. K. Dodd, and J. D. Gibbon from the point of view of the model being integrable, and it possesses infinitely many symmetries [6]. Furthermore, it is proven that the model is one of the three possible Painlevé integrable models and Liouville integrable models [7, 8]. In [9] it has been proven that the

SK equation appears as a conserved flow equation of the well-known Liouville field theory that is widely applied in many physical branches, such as the subcritical strings, conformal field theory, two-dimensional quantum gravity gauge field theory and nonlinear science. The Boussinesq equation can be derived from describing the propagation of gravity waves on the surface of water and in the context of Langmuir waves [10].

The N-plane soliton solutions of the SK equation and the BS equation can be derived by using the bilinear method proposed by Hirota [11]. After having suitable selected some parameters in the N-plane soliton solutions, we obtain breather solutions, the solutions to describe the interaction between a line soliton and a y-periodic soliton, and solutions to express the interaction between two y-periodic solitons. The interactions between a line soliton and a y-periodic soliton for the SK equation, and between two y-periodic solitons for the BS equation are also studied in this paper.

The paper is organized as follows: In Sect. 2 we use the bilinear method to find some special solutions of the SK and BS equations. The interaction between two solitons are discussed in Sect. 3, while a summary and a discussion are given in Section 4.

2. Exact Solutions of two (2+1)-dimensional KdV-type Equations

2.1. Exact Solutions of the (2+1)-dimensional Sawada-Kotera Equation

After the transformation

$$u = 2(\ln f)_{xx}, \tag{3}$$

the bilinear forms of (1) can be written as

$$(D_x^6 + 5D_yD_x^3 - 5D_y^2 + D_xD_t) f \cdot f = 0, \tag{4}$$

where the D operator is defined by [11]

$$D_x^n D_y^m D_t^p f \cdot g \equiv (\partial_x - \partial_{x'})^n (\partial_y - \partial_{y'})^m (\partial_t - \partial_{t'})^p f(x,y,t) \times g(x',y',t') \mid_{(x'=x, y'=y, t'=t)} .$$

It can be easily proved that the N "plane" soliton solution of (4) exists and possesses the standard presentation [12,13]:

$$f = 1 + \sum_{i=1}^{N} \exp(\eta_i) + \sum_{i<j}^{N} a_{ij} \exp(\eta_i + \eta_j) + \sum_{i<j<k}^{N} a_{ij}a_{ik}a_{jk} \exp(\eta_i + \eta_j + \eta_k) + \cdots + \left(\prod_{i<j} a_{ij}\right) \exp\left\{\sum_{i=1}^{N} \eta_i\right\}, \tag{5}$$

$$\eta_i = k_i x + L_i y - \Omega_i t - \eta_i^0, \tag{6}$$

$$\Omega_i = -\frac{k_i^6 + 5L_i k_i^3 - 5L_i^2}{k_i}, \tag{7}$$

$$a_{ij} = -\frac{(k_i - k_j)^6 + 5(L_i - L_j)(k_i - k_j)^3 - 5(L_i - L_j)^2 + (k_i - k_j)(\Omega_i - \Omega_j)}{(k_i + k_j)^6 + 5(L_i + L_j)(k_i + k_j)^3 - 5(L_i + L_j)^2 + (k_i + k_j)(\Omega_i + \Omega_j)}. \tag{8}$$

In (5)–(8), the parameters k_i, L_i, and η_i^0 are real constants relative to the amplitude and phase, respectively, of the i-th soliton.
Although the above soliton is derived for real k_i's, L_i's and η_i's, it still holds for the case that some k_i, L_i and η_i^0 are complex numbers. We obtain a periodic soliton solution of breather type when we consider the case $N = 2$, $k_1 = k_2 = \alpha$, $L_1 = i\delta$, $L_2 = -i\delta$, $\eta_1^0 = \eta_2^0 = 0$. Equation (5) then has the form

$$f = 1 + a_{12} \exp(2\xi) + 2\exp(\xi) \cos(\delta y + 5\delta\alpha^2 t) \tag{9}$$

with

$$\xi = \alpha x - \Omega_r t, \quad \Omega_r = -\frac{\alpha^6 + 5\delta^2}{\alpha}, \quad a_{12} = \frac{\delta^2}{\delta^2 - 3\alpha^6}. \tag{10}$$

Substitution of (9) into (3) gives

$$u = \frac{2\alpha^2 \left(2 + \frac{1}{\sqrt{a_{12}}} \cos(\delta y + 5\delta\alpha^2 t) \cosh(\alpha x - \Omega_r t + \ln\sqrt{a_{12}}\right)}{\left(\cosh(\alpha x - \Omega_r t + \ln\sqrt{a_{12}}) + \frac{1}{\sqrt{a_{12}}} \cos(\delta y + 5\delta\alpha^2 t)\right)^2}. \tag{11}$$

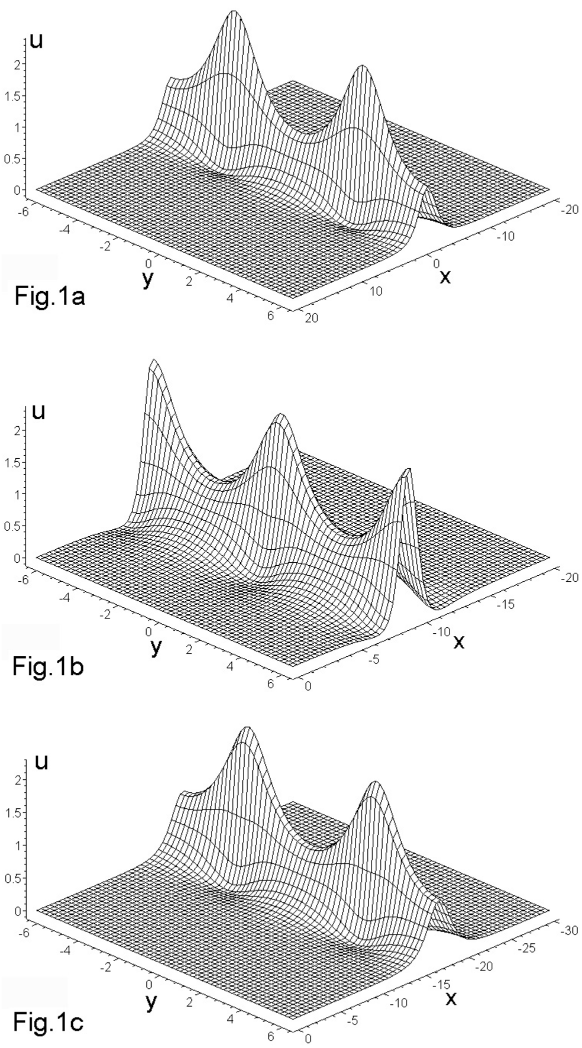


Fig. 1. Plots of one periodic solitons evolution from the SK equation with $(\alpha,\ \delta) = (4/5, 1)$. (a) $t = 0$, (b) $t = 5\pi/16$, (c) $t = 5\pi/8$.

The condition for the existence of a nonsingular solution (11) is

$$a_{12} > 1,$$

from the condition that the denominator in (11) does not vanish. A typical spatial structure of the solution (11) is shown in Figure 1. It represents a sequence of algebraic solitons. We call this solution a periodic soliton. Because the phase factor in the y-direction is a function of t, the internal oscillation is shown this direction. Figs. 1a, 1b and 1c show the phenomenon, $5\delta\alpha^2 t$ is taken as 0, π, 2π, respectively. In (5), taking $N = 3$, $k_1 = k_2 = \alpha_1$, $L_1 = i\delta_1$, $L_2 = -i\delta_1$, where $k_3 = \alpha_2$, $L_3 = 0$, $\eta_1^0 = \eta_2^0 = \eta_3^0 = 0$, we obtain the solution which expresses an interaction between a y-periodic soliton and a line soliton. This solution can be written as

$$u = 2(\log f)_{xx}, \tag{12}$$

$$\begin{aligned} f = {} & 1 + \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_3) \\ & + a_{12}\exp(\eta_1 + \eta_2) + a_{13}\exp(\eta_1 + \eta_3) \\ & + a_{23}\exp(\eta_2 + \eta_3) + a_{12}a_{13}a_{23}\exp(\eta_1 + \eta_2 + \eta_3) \end{aligned} \tag{13}$$

$$\eta_1 = \alpha_1 x + i\delta_1 y - \Omega_1 t, \quad \eta_2 = \alpha_1 x - i\delta_1 y - \Omega_2 t,$$

$$\eta_3 = \alpha_2 x - \Omega_3 t, \quad a_{12} = \frac{\delta_1^2}{\delta_1^2 - 3\alpha_1^6},$$

$$\begin{aligned} a_{23}^* = a_{13} = {} & -\big[(\alpha_1 - \alpha_2)^6 + 5(i\delta_1 - 0)(\alpha_1 - \alpha_2)^3 \\ & - 5(i\delta_1 - 0)^2 + (\alpha_1 - \alpha_2)(\Omega_1 - \Omega_3)\big] \\ & \cdot \big[(\alpha_1 + \alpha_2)^6 + 5(i\delta_1 + 0)(\alpha_1 + \alpha_2)^3 \\ & - 5(i\delta_1 + 0)^2 + (\alpha_1 + \alpha_2)(\Omega_1 + \Omega_3)\big]^{-1}. \end{aligned}$$

After some simple operation we find that the expression (13) possesses the following form:

$$\begin{aligned} f = {} & 1 + a_{12}\exp(2\xi_1) + 2\exp(\xi_1)\cos(\eta) + \exp(\xi_2) \\ & \cdot \big(1 + 2L\exp(\xi_1)\cos(\eta + \phi) \\ & + a_{12}L^2\exp(2\xi_1)\big), \end{aligned} \tag{14}$$

$$\eta = \delta_1 y + 5\delta_1\alpha_1^2 t, \quad \xi_1 = \alpha_1 x - \omega_r t,$$

$$\omega_r = -\frac{\alpha_1^6 + 5\delta_1^2}{\alpha_1}, \tag{15}$$

$$\xi_2 = \alpha_2 x - \omega_2 t = \alpha_2 x + \alpha_2^5 t, \tag{16}$$

$$L\exp(i\phi) = a_{13}, \quad L\exp(-i\phi) = a_{23} = a_{13}^*. \tag{17}$$

If we take $N = 4$, $k_1 = k_2 = \alpha_1$, $L_1 = i\delta_1$, $L_2 = -i\delta_1$, $k_3 = k_4 = \alpha_2$, $L_3 = i\delta_2$, $L_4 = -i\delta_2$, $\eta_1^0 = \eta_2^0 = \eta_3^0 = \eta_4^0 = 0$ in (5) to (8), the solution to describe the interaction between two y-periodic solitons can be obtained, where the interaction should be understood as the collision or scattering between two local structures described by a physical field $u = 2(\log f)_{xx}$ characterizing a soliton. The expression for f is given by

$$f = 1 + a_{12}\exp(2\xi_1) + 2\exp(\xi_1)\cos(\eta_1) + a_{34}\exp(2\xi_2) + 2\exp(\xi_2)\cos(\eta_2)$$
$$+ 2\exp(\xi_1+\xi_2)\left[L_1\cos(\eta_1+\eta_2+\phi_1) + L_2\cos(\eta_2-\eta_1+\phi_2)\right] + 2a_{34}L_1L_2\exp(2\xi_2)\exp(\xi_1)\cos(\eta_1+\phi_1-\phi_2)$$
$$+ 2a_{12}L_1L_2\exp(2\xi_1)\exp(\xi_2)\cos(\eta_2+\phi_1+\phi_2) + a_{12}a_{34}L_1^2L_2^2\exp(2(\xi_1+\xi_2)), \quad (18)$$

with

$$a_{12} = \frac{\delta_1^2}{\delta_1^2 - 3\alpha_1^6}, \quad a_{34} = \frac{\delta_2^2}{\delta_2^2 - 3\alpha_2^6}, \quad (19)$$

$$L_1\exp(i\phi_1) = a_{13} = -\frac{(\alpha_1-\alpha_2)^6 + 5(i\delta_1 - i\delta_2)(\alpha_1-\alpha_2)^3 - 5(i\delta_1 - i\delta_2)^2 + (\alpha_1-\alpha_2)(\Omega_1-\Omega_3)}{(\alpha_1+\alpha_2)^6 + 5(i\delta_1 + i\delta_2)(\alpha_1+\alpha_2)^3 - 5(i\delta_1 + i\delta_2)^2 + (\alpha_1+\alpha_2)(\Omega_1+\Omega_3)}, \quad (20)$$

$$L_2\exp(i\phi_2) = a_{23} = -\frac{(\alpha_1-\alpha_2)^6 + 5(-i\delta_1 - i\delta_2)(\alpha_1-\alpha_2)^3 - 5(-i\delta_1 - i\delta_2)^2 + (\alpha_1-\alpha_2)(\Omega_1-\Omega_3)}{(\alpha_1+\alpha_2)^6 + 5(-i\delta_1 + i\delta_2)(\alpha_1+\alpha_2)^3 - 5(-i\delta_1 + i\delta_2)^2 + (\alpha_1+\alpha_2)(\Omega_1+\Omega_3)}, \quad (21)$$

$$L_1\exp(-i\phi_1) = a_{24}, \quad L_2\exp(-i\phi_2) = a_{14}, \quad (22)$$

$$\xi_i = \alpha_i x - \omega_{ri} t, \quad \omega_{ri} = -\frac{\alpha_i^6 + 5\delta_i^2}{\alpha_i}, \quad (23)$$

$$\eta_i = \delta_i y + 5\alpha_i^2\delta_i t, \quad (i = 1,\ 2). \quad (24)$$

2.2. Exact solutions of the (2+1)-dimensional Boussinesq Equation

Taking the transformation of (3), now (2) has the bilinear form

$$(4D_xD_y - 3D_t^2 + D_x^4 - aD_x^2)f\cdot f = 0. \quad (25)$$

It can be easily proven that the N “plane” soliton solution (25) also exists and possesses the standard presentation. The forms of f and η_i are the same as those in (5) and (6), but Ω_i and a_{ij} are now expressed as

$$4k_iL_i - 3\Omega_i^2 + k_i^4 - ak_i^2 = 0, \quad (26)$$

$$a_{ij} = -\frac{4(k_i-k_j)(L_i-L_j) - 3(\Omega_i-\Omega_j)^2 + (k_i-k_j)^4 - a(k_i-k_j)^2}{4(k_i+k_j)(L_i+L_j) - 3(\Omega_i+\Omega_j)^2 + (k_i+k_j)^4 - a(k_i+k_j)^2}. \quad (27)$$

We obtain the solution to describe the interaction between a line soliton and a y-periodic soliton, if we consider the case: $N = 3$, $k_1 = k_2 = \alpha_1$, $L_1 = i\delta$, $L_2 = -i\delta$, $k_3 = \alpha_2$, $L_3 = 0$, $\eta_1^0 = \eta_2^0 = \eta_3 = 0$. The solution possesses the same form as (12) and (14), however η, ξ_i and the interaction constants a_{ij} have the following forms:

$$\eta = \delta y + \omega_{11}t, \quad \xi_1 = \alpha_1 x + \omega_{10}t, \quad \xi_2 = \alpha_2 x + \omega_2 t, \quad (28)$$

$$\omega_{10} = \frac{\sqrt{A+\sqrt{A^2+B^2}}}{\sqrt{6}}, \quad \omega_{11} = \frac{\sqrt{\sqrt{A^2+B^2}-A}}{\sqrt{6}}, \quad A = \alpha_1^4 - \alpha_1^2, \quad B = 4\alpha_1\delta,$$

$$L\exp(i\phi) = a_{13} = -\frac{4(\alpha_1-\alpha_2)(i\delta-0) - 3(\Omega_1-\Omega_3)^2 + (\alpha_1-\alpha_2)^4 + (\alpha_1-\alpha_2)^2}{4(\alpha_1+\alpha_2)(i\delta+0) - 3(\Omega_1+\Omega_3)^2 + (\alpha_1+\alpha_2)^4 + (\alpha_1+\alpha_2)^2}, \quad (29)$$

$$L\exp(-i\phi) = a_{23} = a_{13}^*, \quad \Omega_1 = \omega_{10} + i\omega_{11}, \quad \Omega_2 = \omega_{10} - i\omega_{11}, \quad \omega_2 = \Omega_3 = \frac{\sqrt{k_3^4 - k_3^2}}{3}. \quad (30)$$

The coupling coefficient a_{12} in eq.(14) is determined by (27) and a in (2) has been taken as 1 for simplicity.

If taking $N = 4$, $k_1 = k_2 = \alpha_1$, $L_1 = i\delta_1$, $L_2 = -i\delta_1$, $k_3 = k_4 = \alpha_2$, $L_3 = i\delta_2$, $L_4 = -i\delta_2$, $-\eta_1^0 = \log(\frac{1}{2}) + i\theta_1'$, $-\eta_2^0 = -\eta_1^{0*}$, $-\eta_3^0 = \log(\frac{1}{2}) + i\theta_2'$, $\eta_4^0 = -\eta_3^{0*}$ and $\exp(i\theta') = -\frac{1}{\alpha^2}\exp(\sigma + i\theta)$ in (5), (6), (26), and (27), the solution which describes the interaction between two y-periodic solitons can be obtained with a field u for the Boussineq equation. The expression for f is then given by

$$\begin{aligned} f = 1 &+ \frac{a_{12}}{4\alpha_1^4}\exp(2\xi_1) - \frac{1}{\alpha_1^2}\exp(\xi_1)\cos(\eta_1) + \frac{a_{34}}{4\alpha_2^4}\exp(2\xi_2) - \frac{1}{\alpha_2^2}\exp(\xi_2)\cos(\eta_2) \\ &+ \frac{L_1}{2\alpha_1^2\alpha_2^2}\exp(\xi_1+\xi_2)\cos(\eta_1+\eta_2+\phi_1) + \frac{L_2}{2\alpha_1^2\alpha_2^2}\exp(\xi_1+\xi_2)\cos(\eta_2-\eta_1+\phi_2) \\ &- \frac{a_{34}L_1L_2}{4\alpha_1^2\alpha_2^4}\exp(2\xi_2+\xi_1)\cos(\eta_1+\phi_1-\phi_2) - \frac{2a_{12}L_1L_2}{4\alpha_1^4\alpha_2^2}\exp(2\xi_1+\xi_2)\cos(\eta_2+\phi_1+\phi_2) \\ &+ \frac{a_{12}a_{34}L_1^2L_2^2}{16\alpha_1^4\alpha_2^4}\exp(2(\xi_1+\xi_2)) \end{aligned} \tag{31}$$

with

$$a_{12} = -\frac{(\alpha_1^4-\alpha_1^2) - \sqrt{(\alpha_1^4-\alpha_1^2)^2+(4\alpha_1\delta_1)^2}}{-7\alpha_1^4+\alpha_1^2-\sqrt{(\alpha_1^4-\alpha_1^2)^2+(4\alpha_1\delta_1)^2}}, \tag{32}$$

$$a_{34} = -\frac{(\alpha_2^4-\alpha_2^2) - \sqrt{(\alpha_2^4-\alpha_2^2)^2+(4\alpha_2\delta_2)^2}}{-7\alpha_2^4+\alpha_2^2-\sqrt{(\alpha_2^4-\alpha_2^2)^2+(4\alpha_2\delta_2)^2}}, \tag{33}$$

$$L_1\exp(i\phi_1) = a_{13} = -\frac{4(\alpha_1-\alpha_2)(i\delta_1-i\delta_2)-3(\Omega_1-\Omega_3)^2+(\alpha_1-\alpha_2)^4+(\alpha_1-\alpha_2)^2}{4(\alpha_1+\alpha_2)(i\delta_1+i\delta_2)-3(\Omega_1+\Omega_3)^2+(\alpha_1+\alpha_2)^4+(\alpha_1+\alpha_2)^2}, \tag{34}$$

$$L_2\exp(i\phi_2) = a_{23} = -\frac{4(\alpha_1-\alpha_2)(-i\delta_1-i\delta_2)-3(\Omega_2-\Omega_3)^2+(\alpha_1-\alpha_2)^4+(\alpha_1-\alpha_2)^2}{4(\alpha_1+\alpha_2)(-i\delta_1+i\delta_2)-3(\Omega_2+\Omega_3)^2+(\alpha_1+\alpha_2)^4+(\alpha_1+\alpha_2)^2}, \tag{35}$$

$$L_1\exp(-i\phi_1) = a_{24}, \quad L_2\exp(-i\phi_2) = a_{14}, \tag{36}$$

$$\xi_i = \alpha_i x + \omega_{i0}t + \sigma_i, \quad \eta_i = \delta_i y + \omega_{ii}t + \theta_i, \tag{37}$$

$$\omega_{i0} = \frac{\sqrt{A_i+\sqrt{A_i^2+B_i^2}}}{\sqrt{6}}, \quad \omega_{ii} = \frac{\sqrt{\sqrt{A_i^2+B_i^2}+A_i}}{\sqrt{6}}, \quad A_i = \alpha_i^4-\alpha_i^2, \quad B_i = 4\alpha_i\delta_i. \tag{38}$$

3. Interactions between two Solitons

3.1. Interaction between a Line Soliton and a y-periodic Soliton for the Sawada-Kodera Equation

In this subsection we discuss the solution (12) with (14) through (17) that describes the superposition of a line soliton and a y-periodic soliton for the SK equation. When we assume that $\alpha_1 > 0$, $\alpha_2 > 0$ and $\omega_2/\alpha_2 > \omega_r/\alpha_1$, we obtain the expressions of a separated line soliton and a y-periodic soliton before and after interactions as follows:

$$f(\xi_1,\eta,\phi,L) = \tag{39}$$

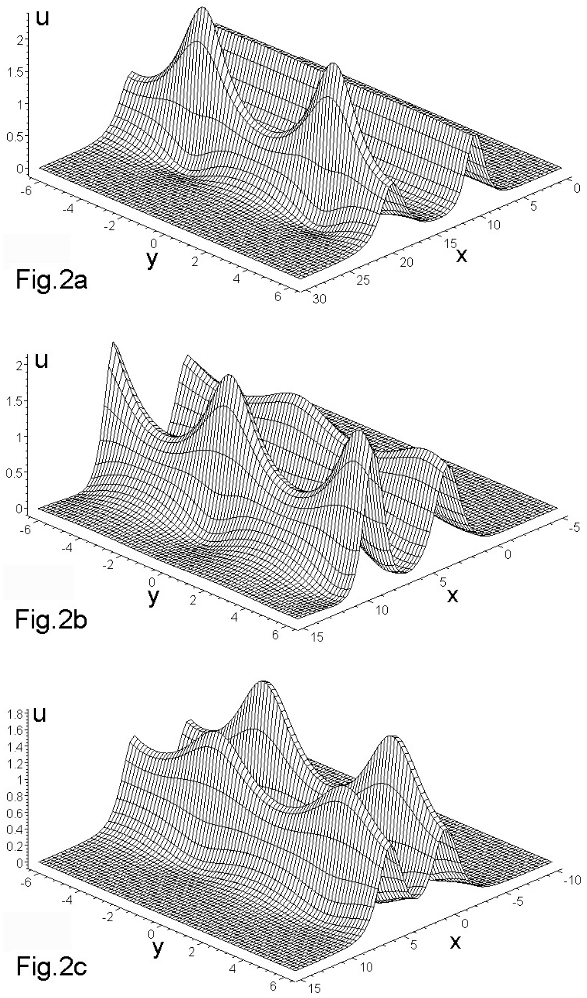
$$\exp(\xi_2)[1+2L\exp(\xi_1)\cos(\eta+\phi)+\mathrm{a}_{12}\mathrm{L}^2\exp(2\xi_1)],$$

$$f(\xi_2) = 1+\exp(\xi_2), \tag{40}$$

and

$$f(\xi_1,\eta) = 1+a_{12}\exp(2\xi_1)+2\exp(\xi_1)\cos(\eta), \tag{41}$$

$$f(\xi_2,L) = a_{12}\exp(2\xi_1)(1+L^2\exp(\xi_2)), \tag{42}$$



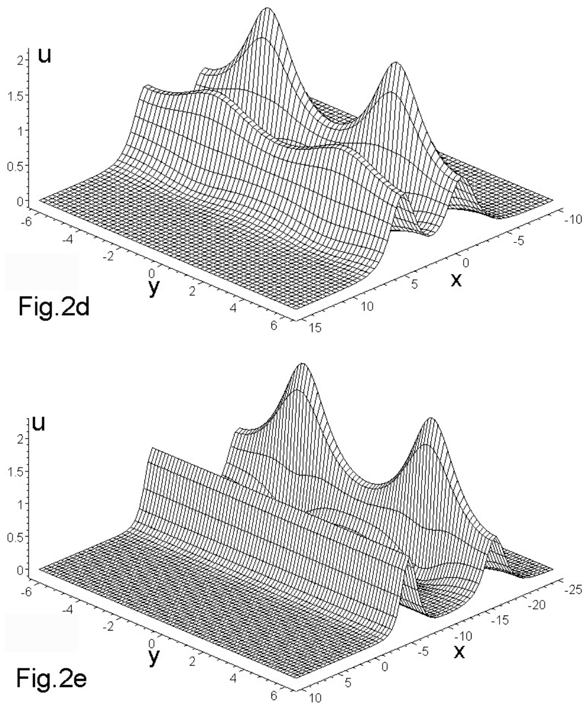


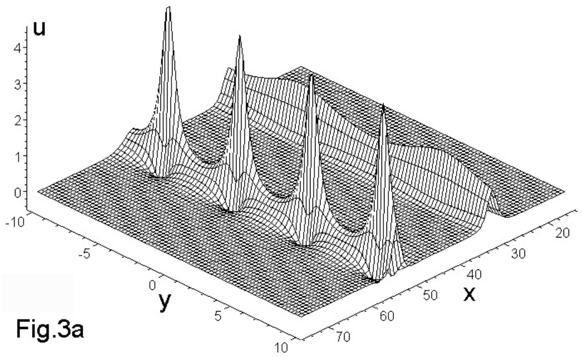
Fig. 2. Plots showing the interaction between a line soliton and a y-periodic soliton for the SK equation with $(\alpha_1,\ \delta_1,\ \alpha_2)$ = (4/5, 1, 3/2). $L = 0.03367$ and $(\omega_2/\alpha_2,\ \omega_r/\alpha_1) = (-81/16, -8.22)$. (a) $t = -5\pi/8$, (b) $t = -0.7$, (c) $t = -0.2$, (d) $t = 0$, (e) $t = 5\pi/8$.

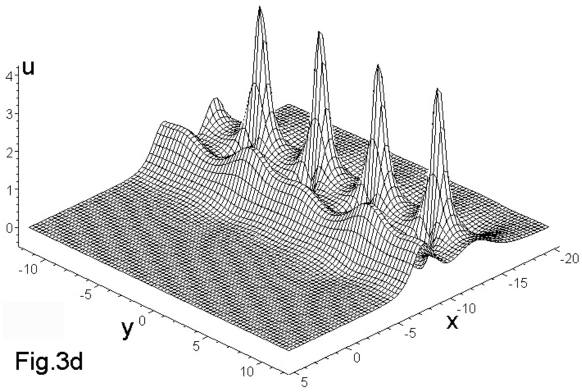
respectively, where the subscripts 1 and 2 denote the coordinates of the y-periodic soliton and the line soliton, respectively. Taking into account that u is unchanged even if f is multiplied by $\exp(ax+b)$ with a and b independent of x, we only have to consider the form of f. In a nutshell we have the following results

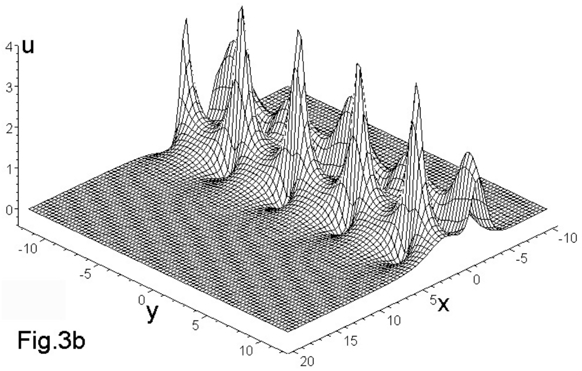
$$[f_1(\xi_1+\Gamma, \eta+\phi), f_2(\xi_2)] \to [f_1(\xi_1, \eta), f_2(\xi_2+2\Gamma)], \tag{43}$$

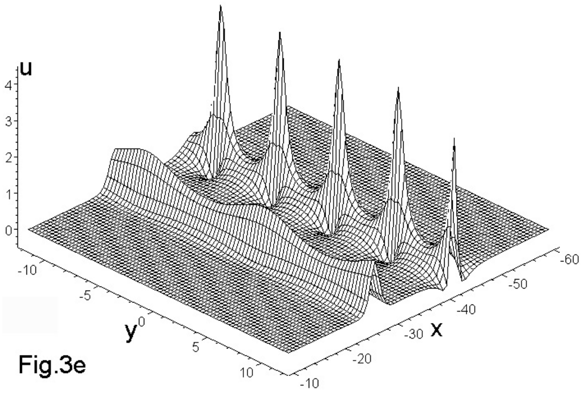
where $\Gamma = \log|L|$. In (43), the left of the arrow shows the phase case of the two solitons at $t \to -\infty$ and the right of the arrow shows the phase case of the two solitons at $t \to \infty$. That is to say, this expression shows that the phase shift due to the interaction is determined by the coefficients (L, ϕ) only. The phase shift in the propagation direction is determined by the magnitude of the coupling coefficient, while that in the transverse direction by the phase ϕ of the coupling coefficient. From (17), one can find that the range of L in (14) is $0 < L < 1$. That is to say, we can only obtain the repulsive interaction between a line soliton and a y-periodic soliton with use of (1). Because η in (14) is a function of the time t, the breathing phenomenon in y-direction is realized.

Figure 2 shows interaction plots between a line soliton and a y-periodic soliton, where $0 < L < 1$, i. e. in the region of a repulsive interaction. The heights of the humps of both solitons are almost the same. As the line soliton approaches the periodic soliton, the frontal part of the line soliton begins to grow, and the formation of the hump on the peak and the other part becomes smaller. At the same time the trough of the periodic soliton grows, as a result, it looks fatter (Figs. 2b, 2c).









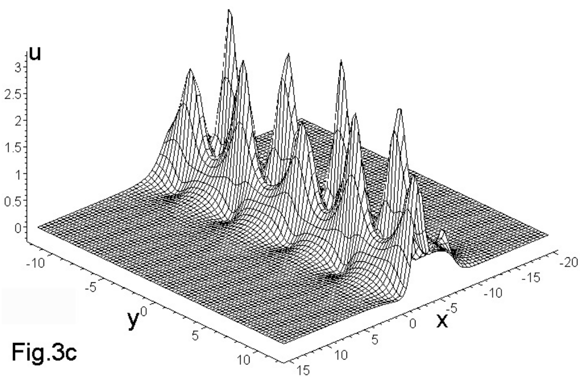


Fig. 3. Plots of the interaction between two y-periodic soliton for the BS equation with $(\alpha_1,\ \delta_1,\ \alpha_2,\ \delta_2)$ = $(2/3, 2/\sqrt{3}, 3/4, 1/\sqrt{3})$; $a = 1$, $L_1 L_2 = 0.2343$; $(\omega_{20}/\alpha_2, \omega_{10}/\alpha_1)$= (0.6674, 1.0324); $(\sigma_1,\ \sigma_2,\ \theta_1,\ \theta_2)$ = (0, 0, 0, 0). (a) $t = -50$, (b) $t = 0$, (c) $t = 5$, (d) $t = 12$, (e) $t = 40$.

While they do not interact at a distance from each other, they seem to exchange some physical quantity related to the physical field u through their tails in the propagating direction. In the y-direction, internal oscillation can be observed in Figure 2.

3.2. *Interaction between two y-periodic Solitons for the BS Equation*

In this subsection we will discuss the interaction between two y-periodic solitons for the BS equation. In (3), assuming that $\alpha_1 > 0$, $\alpha_2 > 0$, and $\omega_{10}/\alpha_1 > \omega_{20}/\alpha_2$, the expressions of two separated y-periodic solitons before and after interaction can be written as:

$$u_1(\xi_1 + \Gamma, \eta_1 + \phi_1 - \phi_2) = 2(\log f_1)_{xx},$$

$$f_1 = \frac{a_{34}}{4\alpha_2^4} \exp(2\xi_2) \Big(1 - \frac{L_1 L_2}{\alpha_1^2} \exp(\xi_1) \cdot \cos(\eta_1 + \phi_1 - \phi_2) + \frac{a_{12} L_1^2 L_2^2}{4\alpha_1^4} \exp(2\xi_1)\Big), \quad (44)$$

$$u_2(\xi_2, \eta_2) = 2(\log f_2)_{xx},$$

$$f_2 = 1 + \frac{a_{34}}{4\alpha_2^4}\exp(2\xi_2) - \frac{1}{\alpha_2^2}\exp(\xi_2)\cos(\eta_2) \tag{45}$$

and

$$u_1(\xi_1,\eta_1) = 2(\log f_1)_{xx},$$

$$f_1 = 1 + \frac{a_{12}}{4\alpha_1^4}\exp(2\xi_1) - \frac{1}{\alpha_1^2}\exp(\xi_1)\cos(\eta_1), \tag{46}$$

$$u_2(\xi_2+\Gamma,\eta_2+\phi_1+\phi_2) = 2(\log f_2)_{xx},$$

$$f_2 = \frac{a_{12}}{4\alpha_1^4}\exp(2\xi_1)\Big(1 - \frac{L_1L_2}{\alpha_2^2}\exp(\xi_2) \cdot \cos(\eta_2+\phi_1+\phi_2) + \frac{a_{34}L_1^2L_2^2}{4\alpha_2^4}\exp(2\xi_2)\Big). \tag{47}$$

For the same reason as that in the subsection 3.1 we have only to consider the form of f. We have the following results:

$$[u_1(\xi_1+\Gamma,\eta_1+\phi_1-\phi_2),u_2(\xi_2,\eta_2)] \rightarrow [u_1(\xi_1,\eta_1),u_2(\xi_2+\Gamma,\eta_2+\phi_1+\phi_2)], \tag{48}$$

where

$$u_i = 2(\log f_i)_{xx}, \quad i = 1,\ 2 \tag{49}$$

is the y-periodic soliton solution and $\Gamma = \log(L_1L_2)$. The meaning of the arrow is the same as that in subsection 3.1. This expression shows that the phase shift due to the interaction is determined only by the product L_1L_2 and by $\{\phi_1,\phi_2\}$. The phase shift in the propagating direction is determined by the product of coupled coefficient L_1 and L_2 while that in the transverse direction by the expression $\{\phi_1-\phi_2\}$ and $\{\phi_1+\phi_2\}$.

Figure 3 displays the interaction plots between two y-periodic solitons, where $0 < L_1L_2 < 1$, i. e., in the region of repulsive interaction. From Fig. 3 we can see that, as two periodic solitons approach each other, two humps of the first soliton begin to move in the transverse direction so as to merge into one hump, and every hump of the second soliton begins to separate in the transverse direction. When this change happens, two solitons keep at a distance from each other. They seem to exchange some physical quantity related to the physical field u through their tails in the propagation direction. Through this interacting the first soliton becomes the second soliton and the second soliton becomes the first soliton. Then they depart from each other, recovering the respective original forms.

4. Summary and Discussion

Using the bilinear approach to find the N plane soliton solution proposed by Hirota, we obtain the solution which describes the interaction between a line soliton and a y-periodic soliton and the solution to express the interaction between two y-periodic solitons for the SK equation and the BS equation. The interactions between a line soliton and a y-periodic soliton for the SK equation and between two y-periodic solitons for the BS equation are discussed both analytically and graphically. The interaction is related to the interaction coefficients (L,ϕ) or $(L_1L_2,\ \phi_1,\ \phi_2)$. The magnitude L or L_1L_2 of the interaction coefficient is related to the phase shift in the propagation direction, and its complex angle ϕ or $(\phi_1,\ \phi_2)$ is related to that in the transverse direction. For the SK equation, the value of L being limited to $0 < L < 1$, we can only obtain the repulsive interaction. For the BS equation we only discuss the repulsive interaction $(0 < L_1L_2 < 1)$. Because η or $(\eta_1,\ \eta_2)$ in these solutions are functions of the time t, internal oscillation phenomena can be observed in these solution.

From Figs. 2 and 3, we can clearly see that the shapes of the two solitons are unchanged after their interaction. This means that there is no exchange of a physical quantity related to u, but there are phase shifts. We call this interaction an elastic interaction. This conclusion is the same as that obtained by us previously [14, 15]. If a multi-soliton solution is characterized by the standard form of Hirota, the interaction between two solitons is completely elastic.

Although we have obtained solutions which describe the interaction between a line soliton and a y-periodic soliton and have expressed their interaction between the two dromions of the ANNV equation and the NNV equation using the separation of variables method introduced before [16, 17], here the SK equation and the BS equation are KdV-type equations with a single component. The abundant soliton structures cannot be obtained by the separation of variables approach. The bilinear method provides a simple password to find exact solutions of this type of models. Whether the solution to describe the interaction between a y-periodic soliton and an algebraic soliton can be obtained via the bilinear approach will be further studied.

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